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# Integral Representation of the Linear Boltzmann Operator for Granular Gas Dynamics with Applications

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**Abstract** We investigate the properties of the collision operator $Q$ associated to the linear Boltzmann equation for *dissipative* hard-spheres arising in granular gas dynamics. We establish that, as in the case of non-dissipative interactions, the gain collision operator is an integral operator whose kernel is made explicit. One deduces from this result a complete picture of the spectrum of $Q$ in an Hilbert space setting, generalizing results from T. Carleman (Publications Scientifiques de l'Institut Mittag-Leffler, vol. 2, 1957) to granular gases. In the same way, we obtain from this integral representation of $Q^+$ that the semigroup in $L^1(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes dv)$ associated to the linear Boltzmann equation for dissipative hard spheres is *honest* generalizing known results from Arlotti (Acta Appl. Math. 23:129–144, 1991).

**Keywords** Granular gas dynamics · Linear Boltzmann equation · Detailed balance law · Spectral theory · $C_0$-semigroup

## 1 Introduction

We deal in this paper with the linear Boltzmann equation for *dissipative interactions* modeling the evolution of a granular gas, undergoing *inelastic collisions* with its underlying medium. Actually, we shall see in the sequel that there is no contrast between the scattering theory of granular gases and that of classical (*elastic*) gases. This may seem quite surprising if one has in mind the fundamental differences that may be emphasized between the *nonlinear kinetic theory* of granular gases and that of classical gases, as briefly recalled in the next lines.

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### 1.1 Granular Gas Dynamics: Linear and Nonlinear Models

Let us begin by recalling the general features of the kinetic description of granular gas dynamics that can be recovered from the monograph [4] or the more mathematically oriented survey [23]. If $f(x,v,t)$ denotes the distribution function of granular particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, then the evolution of $f(x,v,t)$ is governed by the following generalization of Boltzmann equation

$$\partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) = \mathcal{C}(f)(x,v,t), \tag{1.1}$$

with initial condition $f(x,v,0) = f_0(x,v) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes dv)$, where the right-hand side $\mathcal{C}(f)$ models the collision phenomena and depends on the phenomenon we describe.

In the *nonlinear description*, the collision operator $\mathcal{C}(f) =: \mathcal{B}[f,f]$ is a quadratic operator modeling the binary collision phenomena between self-interacting particles. For *hard-spheres* interactions, it reads

$$\mathcal{B}[f,f](v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| \left[ \frac{1}{\epsilon^2} f(x,v_\star,t) f(x,w_\star,t) - f(x,v,t) f(x,w,t) \right] dw dn,$$

where $q$ is the relative velocity, $q = v - w$. The microscopic velocities $(v_\star, w_\star)$ are the pre-collisional velocities of the so-called inverse collision, which results in $(v,w)$ as post-collisional velocities. The main peculiarity of the kinetic description of granular gas is the inelastic character of the microscopic collision mechanism which induces that *part of the total kinetic energy is dissipated*. This energy dissipation might be due to the roughness of the surface or just to a non-perfect restitution and is measured through a restitution coefficient $0 < \epsilon < 1$ (which we assume here to be constant for simplicity, see Remark 1.2). As a consequence, the collision phenomenon is a *non microreversible process*. Generally, we assume that the energy dissipation does not affect the conservation of momentum. Therefore, in the homogeneous setting, i.e. when $f_0(x,v) = f_0(v)$ is independent of the position, the number density of the gas is constant while the bulk velocity is conserved. However, the temperature of the gas

$$\vartheta(t) = \frac{1}{3} \int_{\mathbb{R}^3} |v|^2 f(t,v) dv$$

continuously decreases (*cooling of granular gas*). As a consequence, the stationary state of the inelastic collision operator $\mathcal{B}$ is a given Dirac mass. However, the homogeneous Boltzmann equation for granular gases exhibits self-similar solution (*homogeneous cooling state*) [8, 17]. Note the important contrast with the classical kinetic theory, i.e. when $\epsilon = 1$, for which it is well-known that the steady state of the collision operator is a Maxwellian distribution.

The *linear Boltzmann equation for dissipative interactions* concerns dilute particles (test particles with negligible mutual interactions) immersed in a fluid at thermal equilibrium [14, 16, 22]. The total kinetic energy is dissipated when the dilute particles collide with particles of the host fluid. Such physical models are well-suited to the study of the dynamics of a mixture of impurities in a gas [5, 9] for which the background is in thermodynamic equilibrium and the polluting particles are sufficiently few. We refer the reader to [11] and the survey [10] for more details on the theory of granular gaseous mixtures. Assuming the fluid at thermal equilibrium and neglecting the mutual interactions of both the test and dilute particles, the collision operator $\mathcal{C}(f) = Q(f)$ is a *linear scattering operator* given by

$$Q(f) = \mathcal{B}[f, \mathcal{M}_1] = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| \left[ \frac{1}{\epsilon^2} f(x,v_\star,t) \mathcal{M}_1(w_\star) - f(x,v,t) \mathcal{M}_1(w) \right] dw dn \tag{1.2}$$

where $\mathcal{M}_1$ stands for the distribution function of the host fluid. Note that in such a scattering model, the microscopic masses of the dilute particles $m$ and that of the host particles $m_1$ can be different. We will assume throughout this paper that the distribution function of the host fluid is a given normalized Maxwellian function:

$$\mathcal{M}_1(v) = \left(\frac{m_1}{2\pi\vartheta_1}\right)^{3/2} \exp\left\{-\frac{m_1(v-u_1)^2}{2\vartheta_1}\right\}, \quad v \in \mathbb{R}^3,$$

where $u_1 \in \mathbb{R}^3$ is the given bulk velocity and $\vartheta_1 > 0$ is the given effective temperature of the host fluid. It can be shown in this case that the number density of the dilute gas is the unique conserved macroscopic quantity (as in the elastic case). The temperature is still not conserved but it remains bounded away from zero, which prevents the solution to the linear Boltzmann equation to converge towards a Dirac mass. This strongly contrasts to the nonlinear description and suggests that the linear scattering model associated with granular gases does not contrast too much with the one associated with classical gases.

The first mathematical result in this direction is the following one according to which, as in the classical case, the unique steady state of $Q$ remains Gaussian. The fact that the linear Boltzmann equation still possesses a stationary Maxwellian velocity distribution was first obtained in [16] and we refer to [14] for a complete proof (existence and uniqueness) for hard-spheres model (see also [22] for a version of this result for Maxwell molecules):

**Theorem 1.1** *The Maxwellian velocity distribution*:

$$\mathcal{M}(v) = \left(\frac{m}{2\pi\vartheta^{\#}}\right)^{3/2} \exp\left\{-\frac{m(v-u_1)^2}{2\vartheta^{\#}}\right\}, \quad v \in \mathbb{R}^3,$$

*with* $\vartheta^{\#} = \frac{(1+\epsilon)m}{2m+(3+\epsilon)m_1}\vartheta_1$ *is the unique equilibrium state of* $Q$ *with unit mass.*

*Remark 1.2* Note that, if one does not assume the restitution coefficient $\epsilon$ to be constant (see [4] for the general expression of non-constant restitution coefficient $\epsilon = \epsilon(q)$ in the case, e.g., of visco-elastic spheres) then the nature of the equilibrium state of $Q$ is still an open question: it is not known whether such a steady state is a Maxwellian or not. Consequently, it is still not clear that linear inelastic scattering models behave like elastic ones. For this reason, we shall restrict here our study of the linear Boltzmann equation to a *constant* restitution coefficient. We also point out that, if the distribution function of the host fluid $\mathcal{M}_1$ is not of Gaussian type, the explicit expression of the equilibrium state of $Q$ is an open question to our knowledge.

The existence and uniqueness of such an equilibrium state allows to establish a linear version of the famous $H$-Theorem. Precisely, for any *convex* $C^1$-function $\Phi : \mathbb{R}^+ \to \mathbb{R}$, one can define the associated entropy functional as

$$H_\Phi(f|\mathcal{M}) = \int_{\mathbb{R}^3} \mathcal{M}(v)\Phi\left(\frac{f(v)}{\mathcal{M}(v)}\right)dv. \tag{1.3}$$

**Theorem 1.3** ($H$-Theorem [14, 20]) *Let* $f_0(v)$ *be a space homogeneous distribution function with unit mass and finite entropy, i.e.* $H_\Phi(f_0|\mathcal{M}) < \infty$. *Then,*

$$\frac{d}{dt}H_\Phi(f(t)|\mathcal{M}) \le 0 \quad (t \ge 0), \tag{1.4}$$

*where* $f(t)$ *stands for the* (*unique*) *solution to* (1.1) *in* $L^1(\mathbb{R}^3, dv)$.

Note that such a result is valid for any scattering operator with positive kernel and positive equilibrium [20]. As an important consequence, it can be shown by suitable compactness arguments that any solution to the Boltzmann equation (1.1) (with unit mass) converges towards the Maxwellian equilibrium $\mathcal{M}$. Note also that, for the nonlinear Boltzmann equation for dissipative interactions, the temperature is a trivial Lyapunov functional leading to the convergence of any solution towards a Dirac mass. However, the construction of a Lyapunov functional in the self-similar variables allowing relaxation towards the homogeneous cooling state is still an open question (see, e.g. [17] for related problems).

To summarize, the steady state of the linear collision operator for dissipative interactions has the same nature (a Maxwellian distribution) as the one corresponding to non-dissipative interactions. Moreover, as in the classical case, by virtue of the $H$-Theorem, such a steady state attracts any solution to the space homogeneous Boltzmann equation (1.1). This seems to indicate that *most of the properties of the linear Boltzmann equation for elastic interactions remain valid for inelastic scattering models*. It is the main subject of this paper to make precise and confirm such an indication and the key ingredient will be the derivation of an integral representation of the gain part of the collision operator.

### 1.2 Main Results

The main concern of our paper is the derivation of a suitable representation of the gain part of the collision operator $Q$ as an integral operator with *explicit kernel*. Precisely, the linear collision operator $Q$ can be split into $Q(f) = Q^+(f) - Q^-(f)$, where the gain part is

$$Q^+(f)(v) = \epsilon^{-2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| f(v_\star) \mathcal{M}_1(w_\star) dw dn$$

while

$$Q^-(f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| f(v) \mathcal{M}_1(w) dw dn = \sigma(v) f(v)$$

where the collision frequency $\sigma(v)$ is given by $\sigma(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| \mathcal{M}_1(w) dw dn$. It is well-known that, for non-dissipative interactions, i.e. when $\epsilon = 1$, the gain part $Q^+$ can be written as an integral operator with explicit kernel [6, 15] (see also [7, 12] for similar results for the *linearized* Boltzmann equation). We prove that such a representation is still valid in the dissipative case:

**Theorem 1.4** *If $f \geq 0$ is such that $\sigma(v) f(v) \in L^1(\mathbb{R}^3, dv)$, then*

$$Q^+(f)(v) = \int_{\mathbb{R}^3} k(v, v') f(v') dv'$$

*where the integral kernel $k(v, v')$ can be made explicit* (*see* (2.2)).

Actually, most important is the fact that the integral kernel $k(v, v')$ turns out to be very similar to that obtained in the classical case (see for instance [6, 15]), the only changes standing in some explicit numerical constants. Moreover, as we shall see, the kernel $k(v, v')$ and the Maxwellian distribution $\mathcal{M}$ satisfy the following *detailed balance law*:

$$k(v, v') \mathcal{M}(v') = k(v', v) \mathcal{M}(v), \quad v, v' \in \mathbb{R}^3,$$

that allows us to recover Theorem 1.1 in a direct way. Recall that, in [14], the Gaussian nature of the steady state of $Q$ was obtained by replacing $Q$ by its grazing collision limit.

We derive from these two results some important consequences on the linear Boltzmann equation (1.1) with $\mathcal{C} = Q$. The applications are dealing with the space dependent version of (1.1) as well as with the space homogeneous version of it. The first one concerns the spectral properties of the Boltzmann collision operator in its natural Hilbert space setting.

### 1.3 Spectral Properties of the Boltzmann Operator in $L^2(\mathcal{M}^{-1})$

Applying the above $H$-Theorem 1.3 with the quadratic convex function $\Phi(x) = (x-1)^2$, one sees that a natural function space for the study of the *homogeneous linear Boltzmann equation* is the weighted space $L^2(\mathcal{M}^{-1})$. Now, from Theorem 1.4, it is possible to prove that the gain collision operator $Q^+$ is *compact* in $L^2(\mathcal{M}^{-1})$. This compactness result has important consequences on the structure of the spectrum of $Q$ as an operator in $L^2(\mathcal{M}^{-1})$. Precisely, from Weyl's Theorem, the spectrum of $Q$ in this space is given by the (essential) range of the collision frequency $\sigma(\cdot)$ and of isolated eigenvalues with finite algebraic multiplicities. Since $\lambda = 0$ is a simple eigenvalue of $Q$ (its associated null space is spanned by $\mathcal{M}$), this leads to the existence of a positive spectral gap. In turns, one proves that any solution to the space-homogeneous linear Boltzmann equation (1.1) converges at an exponential rate towards the equilibrium. *These spectral results are technical generalizations of some of the fundamental results of T. Carleman* [6], *but are new in the context of granular gas dynamics*.

### 1.4 Honest Solutions for Hard-Spheres Model

It is easily seen that, for any nonnegative $f$,

$$\int_{\mathbb{R}^3} Q^+(f)(v)dv = \int_{\mathbb{R}^3} \sigma(v) f(v)dv, \tag{1.5}$$

i.e. the collision operator $Q$ is *conservative*. Then, formally, any nonnegative solution $f(x,v,t)$ to (1.1) (with $\mathcal{C} = Q$) should satisfy the following *mass conservation equation*:

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v,t)dxdv = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v,0)dxdv, \quad \forall t > 0. \tag{1.6}$$

It is the main concern of Sect. 4 to prove that such a formal mass conservation property holds true for any nonnegative initial datum $f(x,v,0) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. As well documented in the monograph [3], this is strongly related to the honesty of the $C_0$-semigroup governing equation (1.1). More precisely, if we denote by $T_0$ the streaming operator:

$$\mathcal{D}(T_0) = \{f \in X, v \cdot \nabla_x f \in X\}, \quad T_0 f = -v \cdot \nabla_x f,$$

it is not difficult to see that there exists some extension $G$ of $T_0 + Q$ that generates a $C_0$-semigroup of contractions $(Z(t))_{t\geq 0}$ in $X = L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. According to the so-called "sub-stochastic perturbation" theory, developed in [1, 3, 25], it can be proved that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} Z(t) f(x,v)dxdv = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v)dxdv, \quad \forall f \in X,\ f \geq 0$$

if and only if $G$ is the closure of the full transport operator: $G = \overline{T_0 + Q}$. We show in Sect. 4 that the latter holds. To do so, we shall use the integral representation (Theorem 1.4) in order

to apply some of the results of [1] (see also [3, Chap. 10]) dealing with the classical linear Boltzmann equation.

### 1.5 Organization of the Paper

We derive in Sect. 2 the integral representation of $Q^+$ (Theorem 2.1) as well as some of its immediate consequences concerning the explicit expression of the collision frequency. We also recover Theorem 1.1 through a detailed balance law. Section 3 is devoted to the study of the collision operator $Q$ in the narrow space $L^2(\mathcal{M}^{-1}(v)dv)$ and its spectral consequences. In Sect. 4 we apply the results of Sect. 2 as well as some known facts about the classical linear Boltzmann equation [1, 3] to the honesty of the solutions to the Boltzmann equation for dissipative hard-spheres.

## 2 Integral Representation of the Gain Operator

Let us consider the gain operator for dissipative hard-spheres:

$$Q^+(f)(v) = \epsilon^{-2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| f(v_\star) \mathcal{M}_1(w_\star) dw dn$$

and let $\sigma(v)$ be the corresponding collision frequency:

$$\sigma(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| \mathcal{M}_1(w) dw dn, \quad v \in \mathbb{R}^3.$$

Recall that $\mathcal{M}_1$ is a Maxwellian distribution function with bulk velocity $u_1$ and effective temperature $\vartheta_1$. We recall here the general microscopic description of the pre-collisional velocities $(v_\star, w_\star)$ which result in $(v, w)$ after collision. For a constant restitution coefficient $0 < \epsilon < 1$, one has [4, 23]

$$\begin{cases} v_\star = v - 2\alpha \frac{1-\beta}{1-2\beta} [q \cdot n] n, \\ w_\star = w + 2(1-\alpha) \frac{1-\beta}{1-2\beta} [q \cdot n] n; \end{cases}$$

where $q = v - w$, $\alpha$ is the mass ratio and $\beta$ denotes the inelasticity parameter

$$\alpha = \frac{m_1}{m + m_1}, \qquad \beta = \frac{1-\epsilon}{2}.$$

We show in this section that, as it occurs for the classical Boltzmann equation, $Q^+$ turns out to be an integral operator with explicit kernel. The proof of such a result is based on well-known tools from the linear elastic scattering theory [6, 12, 15] while, in the dissipative case, similar calculations have been performed to derive a Carleman representation of the nonlinear Boltzmann operator in [17].

**Theorem 2.1** (Integral representation of $Q^+$) *For any* $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes \sigma(v) dv)$,

$$Q^+ f(x, v) = \frac{1}{2\epsilon^2 \gamma^2} \int_{\mathbb{R}^3} f(x, v') k(v, v') dv', \tag{2.1}$$

*where*

$$k(v,v') = \left(\frac{m_1}{2\pi\vartheta_1}\right)^{1/2} |v-v'|^{-1} \times \exp\left\{-\frac{m_1}{8\vartheta_1}\left((1+\mu)|v-v'| + \frac{|v-u_1|^2-|v'-u_1|^2}{|v-v'|}\right)^2\right\} \tag{2.2}$$

*with* $\mu = -\frac{2\alpha(1-\beta)-1}{\alpha(1-\beta)} > 0$ *and* $\gamma = \alpha\frac{1-\beta}{1-2\beta}$.

*Proof* The local (in $x$) nature of $Q^+$ is obvious and we can restrict ourselves to prove the result for a function $f \in L^1(\mathbb{R}^3, \sigma(v)dv)$ that does not depend on $x$. Set $\gamma = \alpha\frac{1-\beta}{1-2\beta}$ and $\overline{\gamma} = (1-\alpha)\frac{1-\beta}{1-2\beta}$ so that

$$v_\star = v - 2\gamma[q\cdot n]n \quad \text{and} \quad w_\star = w + 2\overline{\gamma}[q\cdot n]n.$$

The following formula, for smooth $\varphi$:

$$\int_{\mathbb{S}_+} (q\cdot n)\varphi((q\cdot n)n)dn = \frac{|q|}{4}\int_{\mathbb{S}^2}\varphi\left(\frac{q-|q|\sigma}{2}\right)d\sigma = \frac{1}{2}\int_{\mathbb{R}^3}\delta(2x\cdot q + x^2)\varphi(-x/2)dx.$$

applied to

$$\varphi(x) = f(v-2\gamma x)\mathcal{M}_1(w+2\overline{\gamma}x)$$

yields

$$Q^+f(v) = \epsilon^{-2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\delta(2x\cdot q + x^2)f(v+\gamma x)\mathcal{M}_1(w-\overline{\gamma}x)dwdx.$$

The change of variables $x \mapsto v' = v + \gamma x$ leads to

$$Q^+f(v) = \epsilon^{-2}\gamma^{-3}\int_{\mathbb{R}^3\times\mathbb{R}^3}\delta(2\gamma^{-1}(v'-v)\cdot q + \gamma^{-2}|v'-v|^2)f(v')\mathcal{M}_1\left(w - \frac{\overline{\gamma}}{\gamma}(v'-v)\right)dwdv'.$$

Now, keeping $v$ and $v'$ fixed, we perform the change of variables $w \mapsto w' = w - \frac{\overline{\gamma}}{\gamma}(v'-v)$, which leads to

$$Q^+f(v) = \epsilon^{-2}\gamma^{-3}\int_{\mathbb{R}^3\times\mathbb{R}^3}\delta\left(2\gamma^{-1}(v'-v)\cdot\left[v-w'-\frac{\overline{\gamma}}{\gamma}(v'-v)\right] + \gamma^{-2}|v'-v|^2\right) \times f(v')\mathcal{M}_1(w')dw'dv'.$$

Writing $w' = v + \lambda_1 n + V_2$ with $\lambda_1 = (w'-v)\cdot n \in \mathbb{R}$, $n = (v'-v)/|v'-v|$ and $V_2\cdot n = 0$, we get, noting that $dw' = dV_2 d\lambda_1$,

$$Q^+f(v) = \epsilon^{-2}\gamma^{-3}\int_{\mathbb{R}^3} f(v')dv'\int_{\mathbb{R}} d\lambda_1\int_{V_2\cdot n=0}\mathcal{M}_1(v+V_2+\lambda_1 n)dV_2 \times \delta(\gamma^{-2}|v'-v|^2 - 2\overline{\gamma}\gamma^{-2}|v'-v|^2 - 2\gamma^{-1}\lambda_1|v'-v|).$$

Thanks to the change of variables $\lambda_1 \mapsto 2\gamma^{-1}|v'-v|\lambda_1$, one can evaluate the Dirac mass as

$$\int_{\mathbb{R}} \delta(\gamma^{-2}|v'-v|^2 - 2\overline{\gamma}\gamma^{-2}|v'-v|^2 - 2\gamma^{-1}\lambda_1|v'-v|)\mathcal{M}_1(v+V_2+\lambda_1 n)d\lambda_1$$
$$= \frac{\gamma}{2|v'-v|}\mathcal{M}_1\left(v+V_2+\frac{1-2\overline{\gamma}}{2\gamma}(v'-v)\right)$$

where we used that $n=(v'-v)/|v'-v|$. Consequently,

$$Q^+ f(v) = \frac{1}{2\epsilon^2\gamma^2}\int_{\mathbb{R}^3} k(v,v')f(v')dv'$$

where

$$k(v,v') = \frac{1}{|v-v'|}\int_{V_2\cdot(v'-v)=0} \mathcal{M}_1\left(v+V_2+\frac{1-2\overline{\gamma}}{2\gamma}(v'-v)\right)dV_2.$$

It remains now to explicit $k(v,v')$. We will use the approach of [15]. Let us assume $v, v'$ to be fixed. Let $\mathsf{P}$ be the hyperplan orthogonal to $(v'-v)$. For any $V_2 \in \mathsf{P}$, set

$$z = v + \frac{1-2\overline{\gamma}}{2\gamma}(v'-v) + V_2 - u_1$$

so that

$$k(v,v') = \left(\frac{\varrho_1}{\pi}\right)^{3/2}|v-v'|^{-1}\int_{V_2\in\mathsf{P}} \exp\{-\varrho_1 z^2\}dV_2$$

where $\varrho_1 = \frac{m_1}{2\vartheta_1}$. Denoting for simplicity $u = \frac{v+v'}{2} - u_1$ and $\mu = -\frac{1-2\overline{\gamma}}{\gamma}$, one has

$$z^2 = \left(u + \frac{v-v'}{2} + \frac{\mu}{2}(v-v') + V_2\right)^2$$
$$= |u+V_2|^2 + \frac{(1+\mu)^2}{4}|v-v'|^2 + \frac{1+\mu}{2}(|v-u_1|^2 - |v'-u_1|^2)$$

where we used the fact that $V_2$ is orthogonal to $(v'-v)$. Splitting $u$ as

$$u = u_0 + u_\perp$$

where $u_0$ is parallel to $v-v'$ while $u_\perp$ is orthogonal to $v-v'$ (i.e. $u_\perp \in \mathsf{P}$), we see that

$$|u+V_2|^2 = |u_0|^2 + |u_\perp + V_2|^2 \quad \text{and} \quad |u_0|^2 = \frac{[|v-u_1|^2 - |v'-u_1|^2]^2}{4|v-v'|^2},$$

so that

$$k(v,v') = |v-v'|^{-1}\left(\frac{\varrho_1}{\pi}\right)^{3/2}\int_{\mathsf{P}} \exp(-\varrho_1|u_\perp+V_2|^2)dV_2$$
$$\times \exp\left\{-\frac{\varrho_1}{4}\left((1+\mu)^2|v-v'|^2 + 2(1+\mu)(|v-u_1|^2 - |v'-u_1|^2)\right.\right.$$
$$\left.\left. + \frac{[|v-u_1|^2 - |v'-u_1|^2]^2}{|v-v'|^2}\right)\right\}.$$

Finally, since $u_\perp \in \mathsf{P}$,

$$\int_{\mathsf{P}} \exp(-\varrho_1 |u_\perp + V_2|^2) dV_2 = \int_{\mathbb{R}^2} \exp(-\varrho_1 x^2) dx = \frac{\pi}{\varrho_1},$$

one obtains the desired expression for $k(v, v')$. □

The very important fact to be noticed out is that the expression of $k(v, v')$ is very similar to that one obtains in the elastic case [15], the only change being the expression of the constant $\mu$. In particular, in the elastic case $\epsilon = 1$, we recover the expression of the kernel obtained in [6] for particles of same mass (i.e. $m = m_1$) and in [15] for particles with different masses.

Another fundamental property of the kernel $k(v, v')$ is that it allows us to recover the steady state of $Q$ through some *microscopic detailed balance law*. Precisely,

**Theorem 2.2** *With the notations of the Theorem 2.1, the following* detailed balance law:

$$k(v, v') \exp\left\{-\frac{m_1}{2\vartheta_1}(1+\mu)(v' - u_1)^2\right\} = k(v', v) \exp\left\{-\frac{m_1}{2\vartheta_1}(1+\mu)(v - u_1)^2\right\}, \quad (2.3)$$

*holds for any $v, v' \in \mathbb{R}^3$. As a consequence, the Maxwellian velocity distribution*:

$$\mathcal{M}(v) = \left(\frac{m}{2\pi\vartheta^\#}\right)^{3/2} \exp\left\{-\frac{m(v-u_1)^2}{2\vartheta^\#}\right\} \quad v \in \mathbb{R}^3,$$

*with $\vartheta^\# = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)}\vartheta_1$ is the unique equilibrium state of $Q$ with unit mass.*

*Proof* According to (2.2), it is easily seen that

$$k(v', v) = k(v, v') \exp\left\{\frac{m_1}{2\vartheta_1}(1+\mu)(|v-u_1|^2 - |v'-u_1|^2)\right\}, \quad v, v' \in \mathbb{R}^3$$

which is nothing but (2.3). Now, writing $\frac{m_1}{2\vartheta_1}(1+\mu) = \frac{m}{2\vartheta^\sharp}$, straightforward calculations lead to the desired expression for the equilibrium temperature $\vartheta^\sharp$. The fact that $\mathcal{M}$ is an equilibrium solution with unit mass follows then from the fact that

$$Q(\mathcal{M})(v) = \int_{\mathbb{R}^3} k(v, v')\mathcal{M}(v')dv' - \sigma(v)\mathcal{M}(v) = \int_{\mathbb{R}^3} [k(v, v')\mathcal{M}(v') - k(v', v)\mathcal{M}(v)]dv'$$

and from the detailed balance law (2.3). To prove that the steady state is unique, we adopt the strategy of [21, Theorem 1]. Precisely, consider the equation

$$\sigma(v) f(v) = Q^+ f(v), \quad \forall v \in \mathbb{R}^3 \quad (2.4)$$

which admits at least the solution $f = \mathcal{M}$. Since $\sigma(v)$ does not vanish, any solution $f$ to (2.4) is such that

$$f(v) = \frac{1}{\sigma(v)} Q^+(f)(v), \quad \forall v \in \mathbb{R}^3.$$

Since $Q^+$ is an integral operator with *nonnegative* kernel, it is clear that $\sigma(v)|f(v)| \leq Q^+(|f|)(v)$ for any $v \in \mathbb{R}^3$. Now, from the positivity of both $\sigma$ and $Q^+$, one sees that the conservation of mass (1.5) reads:

$$\|\sigma f\|_X = \|\sigma\,|f|\,\|_X = \|Q^+(|f|)\|_X.$$

This shows that, actually, $|Q^+(f)(v)| = \sigma(v)|f(v)| = Q^+(|f|)(v)$ for any $v \in \mathbb{R}^3$. Again, since $Q^+$ is a positive operator, one obtains that

$$f = \pm|f|.$$

Now, assume that (2.4) admits two solutions $f_1$, $f_2$ with $\int_{\mathbb{R}^3} f_1(v)dv = \int_{\mathbb{R}^3} f_2(v)dv = 1$. Then, $f_1 - f_2$ is again a solution to (2.4) so that, $f_1 - f_2 = \pm|f_1 - f_2|$. Thus,

$$\pm\int_{\mathbb{R}^3} |f_1(v) - f_2(v)|dv = \int_{\mathbb{R}^3} f_1(v)dv - \int_{\mathbb{R}^3} f_2(v)dv = 0$$

and the uniqueness follows. □

The above result allows to derive the explicit expression of the collision frequency $\sigma(v)$:

**Corollary 2.3** *The collision frequency $\sigma(v)$ for dissipative hard-spheres interactions is given by*

$$\sigma(v) = \frac{2\pi}{(2+\mu)^2}\sqrt{\frac{m_1}{2\pi\vartheta_1}}\left\{\frac{4\vartheta_1}{m_1}\exp\left(-\frac{m_1}{2\vartheta_1}|v-u_1|^2\right)\right.$$
$$\left. + \left(2|v-u_1| + \frac{2\vartheta_1}{m_1|v-u_1|}\right)\int_0^{2|v-u_1|}\exp\left(-\frac{m_1}{8\vartheta_1}t^2\right)dt\right\}. \tag{2.5}$$

*Consequently, there exist positive constants $\nu_0$, $\nu_1$ such that*

$$\nu_0(1+|v-u_1|) \leq \sigma(v) \leq \nu_1(1+|v-u_1|), \quad \forall v \in \mathbb{R}^3.$$

*Proof* Set $C = \sqrt{\frac{m_1}{2\pi\vartheta_1}}$. Noting that $\sigma(v) = \int_{\mathbb{R}^3} k(v', v)dv'$ for any $v \in \mathbb{R}^3$, one has, with the change of variable $z = v' - v$, in a polar coordinate system in which $v$ lies on the third axis

$$\sigma(v) = C\int_{\mathbb{R}^3}\exp\left\{-\frac{m_1}{8\vartheta_1}\left((1+\mu)|z| - \frac{|v-u_1|^2 - |z+v-u_1|^2}{|z|}\right)^2\right\}|z|^{-1}dz$$
$$= 2\pi C\int_0^\infty d\varrho\int_0^\pi \varrho\exp\left\{-\frac{m_1}{8\vartheta_1}((2+\mu)\varrho + 2|v-u_1|\cos\varphi)^2\right\}\sin\varphi\, d\varphi.$$

The computation of this last integral leads to the desired expression for $\sigma(v)$. The estimates are then straightforward [15]. □

## 3 Application to the Boltzmann Operator in $L^2(\mathcal{M}^{-1})$

We investigate in this section the properties of the Boltzmann operator $Q$ in the weighted space

$$\mathcal{H} = L^2(\mathbb{R}^3; \mathcal{M}^{-1}(v)dv).$$

We shall denote by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ the inner product in $\mathcal{H}$. The introduction of such an Hilbert space setting is motivated by the application of the $H$-Theorem 1.3 with the convex function $\Phi(x)=(x-1)^2$. In this case, one sees that, if $f_0\geq 0$ is a space homogeneous initial distribution such that

$$\int_{\mathbb{R}^3} f_0(v)dv=1,\qquad \int_{\mathbb{R}^3}|f_0(v)|^2\mathcal{M}^{-1}(v)dv<\infty,$$

then any solution $f(t,v)$ to the space homogeneous equation

$$\partial_t f(t,v)=Q(f)(t,v),\quad f(0,v)=f_0(v)\in\mathcal{H}, \tag{3.1}$$

satisfies the following estimate:

$$\frac{d}{dt}\int_{\mathbb{R}^3}|f(t,v)-\mathcal{M}(v)|^2\mathcal{M}(v)^{-1}dv\leq 0,\quad t\geq 0.$$

In other words, the mapping $t\longmapsto\|f(t,\cdot)-\mathcal{M}\|_{\mathcal{H}}$ is *nonincreasing*. For these reasons, the study of the properties of the collision operator $Q$ in $\mathcal{H}$ is of particular relevance for the asymptotic behavior of the solution to

$$\partial_t f(t,v)=Q(f)(t,v),\quad f(0,v)\in\mathcal{H}. \tag{3.2}$$

The material of this section borrows some techniques already employed by T. Carleman [6] in the study of non-dissipative gas dynamics (see also, e.g. [7] or [12] for similar results in the context of the *linearized* Boltzmann equation). Let $\mathcal{L}$ be the realization of the operator $Q$ in $\mathcal{H}$, i.e.

$$\mathcal{D}(\mathcal{L})=\left\{f\in\mathcal{H};\int_R|f(v)|^2\sigma^2(v)\mathcal{M}^{-1}(v)dv<\infty\right\}$$

and, for any $f\in\mathcal{D}(\mathcal{L})$, $\mathcal{L}f(v)=Q(f)(v)$ is given by (1.2). As previously, one can use the following splitting of $\mathcal{L}$ as a gain operator and a loss (multiplication) operator, $\mathcal{L}=\mathcal{L}^+-\mathcal{L}^-$ with

$$\mathcal{L}^+(f)(v)=\int_{\mathbb{R}^3}k(v,v')f(v')dv'\quad\text{and}\quad\mathcal{L}^-(f)=\sigma(v)f(v),\quad f\in\mathcal{D}(\mathcal{L}).$$

We shall show, as in the classical case, that $\mathcal{L}^+$ is actually a bounded operator in $\mathcal{H}$. Precisely, let $\mathcal{J}$ define the natural bijection operator from $L^2(\mathbb{R}^3,dv)$ to $\mathcal{H}$:

$$\begin{cases}\mathcal{J}:L^2(\mathbb{R}^3,dv)\longrightarrow\mathcal{H}\\ f\longmapsto\mathcal{J}f(v)=\mathcal{M}^{1/2}(v)f(v).\end{cases}$$

It is clear that $\mathcal{J}$ is a bounded bijective operator whose inverse is given by

$$\mathcal{J}^{-1}g(v)=\mathcal{M}^{-1/2}(v)g(v)\in L^2(\mathbb{R}^3,dv),\quad\forall g\in\mathcal{H}.$$

Now, let us define

$$G(v,v')=\mathcal{M}^{-1/2}(v)k(v,v')\mathcal{M}^{1/2}(v'),\quad v,v'\in\mathbb{R}^3,$$

i.e.

$$G(v,v') = \left(\frac{m_1}{2\pi\vartheta_1}\right)^{1/2} |v-v'|^{-1} \times \exp\left\{-\frac{m_1}{8\vartheta_1}\left((1+\mu)^2|v-v'|^2 + \frac{(|v-u_1|^2-|v'-u_1|^2)^2}{|v-v'|^2}\right)\right\}. \quad (3.3)$$

From the detailed balance law (2.3), one easily checks that $G(v,v') = G(v',v)$ for any $v, v' \in \mathbb{R}^3$. Therefore, defining $\mathcal{G}$ as the integral operator in $L^2(\mathbb{R}^3, dv)$ with kernel $G(v,v')$, i.e.

$$\mathcal{G}f(v) = \int_{\mathbb{R}^3} G(v,v')f(v')dv',$$

one can prove the following:

**Proposition 3.1** *$\mathcal{G}$ is a bounded symmetric operator in $L^2(\mathbb{R}^3, dv)$ and $\mathcal{L}^+ = \mathcal{J}\mathcal{G}\mathcal{J}^{-1}$. Consequently, $\mathcal{L}^+$ is a bounded symmetric operator in $\mathcal{H}$.*

*Proof* It is clear that $\mathcal{G}$ is symmetric since $G(v,v') = G(v',v)$. Now, to prove the boundedness of $\mathcal{G}$, one adopts a strategy already used in the non-dissipative case by T. Carleman [6, p. 75] and shows easily that

$$C := \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} G(v,v')dv' < \infty.$$

Since $G(\cdot,\cdot)$ is symmetric, one also has $\sup_{v'\in\mathbb{R}^3} \int_{\mathbb{R}^3} G(v,v')dv = C < \infty$. Denoting by $\langle\cdot,\cdot\rangle$ the usual inner product of $L^2(\mathbb{R}^3, dv)$, one deduces from Cauchy–Schwarz identity,

$$\langle \mathcal{G}f, g\rangle \le \frac{C}{2}\left(\int_{\mathbb{R}^3} |f(v)|^2 dv + \int_{\mathbb{R}^3} |g(v')|^2 dv'\right), \quad \forall f, g \in L^2(\mathbb{R}^3, dv),$$

which leads to the boundedness of $\mathcal{G}$. Now, since $G(v,v') = \mathcal{M}^{-1/2}(v)k(v,v')\mathcal{M}^{1/2}(v')$ for any $v, v' \in \mathbb{R}^3$, one gets easily that $\mathcal{L}^+ = \mathcal{J}\mathcal{G}\mathcal{J}^{-1}$ and the conclusion follows. □

In Proposition 3.1, we proved that the gain operator $\mathcal{L}^+$ is bounded in $\mathcal{H}$, i.e. $\mathcal{L}^+ \in \mathcal{B}(\mathcal{H})$. Actually, we have much better and it is possible, as in the non-dissipative case, to prove that $\mathcal{L}^+$ is a compact operator in $\mathcal{H}$. Precisely, the following lemma is a direct consequence of Theorem 2.1 and similar calculations valid for the non-dissipative case [6, p. 70–75]. However, we give a detailed proof of it since the known similar results by T. Carleman are all dealing with the case $m = m_1$ and $\epsilon = 1$. It has to be checked that taking account the parameters $m \neq m_1$ and $\epsilon < 1$ does not lead to supplementary difficulty (see Remark 3.8 where the role of $\epsilon \neq 1$ does not allow to adapt *mutatis mutandis* a result valid in the elastic case).

**Lemma 3.2** *For any $0 < p < 3$ and any $q \ge 0$, there exists $C(p,q) > 0$ such that*

$$\int_{\mathbb{R}^3} |G(v,v')|^p \frac{dv'}{(1+|v'-u_1|)^q} \le \frac{C(p,q)}{(1+|v-u_1|)^{q+1}}, \quad \forall v \in \mathbb{R}^3.$$

*Proof* The proof is a technical generalization of a similar result due to T. Carleman [6] in the classical case (i.e. when $m = m_1$ and $\epsilon = 1$). Let us fix $0 < p < 3$ and $q \geq 0$ and set

$$I(v) = \int_{\mathbb{R}^3} |G(v, v')|^p \frac{dv'}{(1 + |v' - u_1|)^q}.$$

Then, one sees easily that

$$I(v) = 2\pi \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \int_0^\pi \sin\varphi d\varphi \times \int_0^\infty \varrho^{2-p} \frac{\exp\{-\frac{m_1 p}{8\vartheta_1}((1+\mu)^2\varrho^2 + (\varrho + 2|v - u_1|\cos\varphi)^2)\}}{(1 + \sqrt{\varrho^2 + |v - u_1|^2 + 2\varrho|v - u_1|\cos\varphi})^q} d\varrho.$$

Note that, since $0 < p < 3$,

$$\sup_{v\in\mathbb{R}^3} I(v) \leq 4\pi \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \int_0^\infty \varrho^{2-p} \exp\left\{-\frac{m_1 p}{8\vartheta_1}(1+\mu)^2\varrho^2\right\} d\varrho < \infty. \tag{3.4}$$

Performing the change of variable $x = \varrho/|v - u_1| + 2\cos\varphi$, $y = \varrho/|v - u_1|$, one has $(x, y) \in \Omega$ where

$$\Omega = \{(x, y) \in \mathbb{R}^2;\ y > 0,\ |x - y| \leq 2\}$$

and

$$I(v) = \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \pi|v - u_1|^{3-p} \times \int_\Omega \exp\left\{-\frac{m_1 p|v - u_1|^2}{8\vartheta_1}((1+\mu)^2 y^2 + x^2)\right\} \frac{dxdy}{y^{p-2}(1 + |v - u_1|\sqrt{1 + xy})^q}.$$

We split $\Omega$ into $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1$ is the half-ellipse

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 : y > 0,\ (1+\mu)^2 y^2 + x^2 < 1/4\} \quad \text{while} \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Note that, since $1 + \mu \geq 1$, one has $\Omega_1 \subset \Omega$. One defines correspondingly $I_1(v)$ and $I_2(v)$ as the above integral over $\Omega_1$ and $\Omega_2$ respectively. One notes first that, if $(x, y) \in \Omega_1$ then $xy > -\frac{1}{8(1+\mu)}$ so that

$$I_1(v) \leq \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \frac{\pi|v - u_1|^{3-p}}{(1 + a|v - u_1|)^q} \int_{\Omega_1} \exp\left\{-\frac{m_1 p|v - u_1|^2}{8\vartheta_1}((1+\mu)^2 y^2 + x^2)\right\} \frac{dxdy}{y^{p-2}}$$

where $a = \sqrt{1 - \frac{1}{8(1+\mu)}}$, $0 < a < 1$. Letting $R = (\frac{m_1 p}{8\vartheta_1})^{1/2}$ and setting $t = R|v - u_1|x$, $u = R(1+\mu)|v - u_1|y$, it is easy to check that $-R|v - u_1|/2 \leq t \leq R|v - u_1|/2$, while $0 \leq u \leq R|v - u_1|/2$, so that

$$I_1(v) \leq \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \frac{\pi R^{p-4}(1+\mu)^{p-3}}{|v - u_1|(1 + a|v - u_1|)^q} \int_{\mathbb{R}} dt \int_0^\infty \frac{\exp\{-(t^2 + u^2)\}}{u^{p-2}} du.$$

Thus, there exists a constant $C_1(p,q) > 0$ such that

$$I_1(v) \le \frac{C_1(p,q)}{|v-u_1|(1+a|v-u_1|)^q}, \quad \forall v \in \mathbb{R}^3. \tag{3.5}$$

Let us now deal with $I_2(v)$. Arguing as above,

$$I_2(v) = \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \pi|v-u_1|^{3-p} \int_{\Omega_2} \exp\left\{-\frac{R^2|v-u_1|^2}{2}((1+\mu)^2y^2+x^2)\right\}$$
$$\times \frac{\exp\{-\frac{R^2|v-u_1|^2}{2}((1+\mu)^2y^2+x^2)\}}{y^{p-2}(1+|v-u_1|\sqrt{1+xy})^q} dxdy.$$

Clearly, since $(1+\mu)^2y^2+x^2 > 1/4$ for any $(x,y) \in \Omega_2$, then

$$\begin{aligned} I_2(v) &\le \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \pi|v-u_1|^{3-p} \int_{\Omega_2} \exp\left(-\frac{R^2|v-u_1|^2}{8}\right) \\ &\quad \times \exp\left\{-\frac{R^2|v-u_1|^2}{2}((1+\mu)^2y^2+x^2)\right\} \frac{dxdy}{y^{p-2}} \\ &\le \left(\frac{m_1}{2\pi\vartheta_1}\right)^{p/2} \pi|v-u_1|^{3-p} \exp\left(-\frac{R^2|v-u_1|^2}{8}\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{R^2|v-u_1|^2}{2}(1+\mu)^2y^2\right) \frac{dy}{y^{p-2}} \int_{y-2}^{y+2} dx. \end{aligned}$$

Hence, there is some constant $C_2(p,q)$ such that

$$I_2(v) \le C_2(p,q) \exp\left(-\frac{R^2|v-u_1|^2}{8}\right), \quad v \in \mathbb{R}^3. \tag{3.6}$$

Combining (3.5) and (3.6), one sees that

$$I(v) \le \frac{C_1(p,q)}{|v-u_1|(1+a|v-u_1|)^q} + C_2(p,q)\exp\left(-\frac{R^2|v-u_1|^2}{8}\right), \quad v \in \mathbb{R}^3.$$

According to (3.4), $\limsup_{|v-u_1|\to 0} I(v) < \infty$, from which we get the conclusion. □

*Remark 3.3* Note that the above Lemma can be extended to more general collision kernels (including long-range interactions) following the lines of the recent results [18] dealing with the elastic case.

From the above Lemma, one has the following compactness result:

**Proposition 3.4** *$\mathcal{G}$ is compact in $L^2(\mathbb{R}^3, dv)$. Consequently, $\mathcal{L}^+$ is a compact operator in $\mathcal{H}$.*

*Proof* Applying arguments already used in [6], the above Lemma implies that the third iterate of $\mathcal{G}$ is an Hilbert–Schmidt operator in $L^2(\mathbb{R}^3, dv)$, i.e. the kernel of $\mathcal{G}^3$ is square summable over $\mathbb{R}^3 \times \mathbb{R}^3$. The compactness of $\mathcal{G}$ follows then from standard arguments and that of $\mathcal{L}^+$ is deduced from the identity $\mathcal{L}^+ = \mathcal{J}\mathcal{G}\mathcal{J}^{-1}$ (see Proposition 3.1). □

The following, which generalizes a known result from classical kinetic theory, proves that $\mathcal{L}$ is a negative symmetric operator in $\mathcal{H}$:

**Proposition 3.5** *The operator* $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ *is a negative self-adjoint operator of* $\mathcal{H}$. *Precisely*,

$$\langle \mathcal{L}f, f\rangle_{\mathcal{H}} = -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} k(v,v')\mathcal{M}(v')[\mathcal{M}^{-1}(v)f(v) - f(v')\mathcal{M}^{-1}(v')]^2 dv dv' \leq 0$$

*for any* $f \in \mathcal{D}(\mathcal{L})$.

*Proof* The fact that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is self-adjoint is a direct consequence of Proposition 3.1 since $\mathcal{L}^-$ is clearly symmetric. Now, it is a classical feature, from the detailed balance law (2.3), that

$$\langle \mathcal{L}f, f\rangle_{\mathcal{H}} = \int_{\mathbb{R}^3\times\mathbb{R}^3} k(v,v')\mathcal{M}(v')[\mathcal{M}^{-1}(v')f(v') - f(v)\mathcal{M}^{-1}(v)]f(v)\mathcal{M}^{-1}(v) dv dv'.$$

Exchanging $v$ and $v'$ and using again the detailed balance law (2.3), one sees that

$$\langle \mathcal{L}f, f\rangle_{\mathcal{H}} = \int_{\mathbb{R}^3\times\mathbb{R}^3} k(v,v')\mathcal{M}(v')[\mathcal{M}^{-1}(v)f(v) - f(v')\mathcal{M}^{-1}(v')]f(v')\mathcal{M}^{-1}(v') dv dv'$$

so that, taking the mean of the two quantities,

$$\langle \mathcal{L}f, f\rangle_{\mathcal{H}} = -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} k(v,v')\mathcal{M}(v')[\mathcal{M}^{-1}(v)f(v) - f(v')\mathcal{M}^{-1}(v')]^2 dv dv' \leq 0$$

which ends the proof. □

*Remark 3.6* From the above result, the spectrum $\mathfrak{S}(\mathcal{L})$ of $\mathcal{L}$ lies in $\mathbb{R}_-$, i.e. $\mathfrak{S}(\mathcal{L}) \subset (-\infty, 0]$. It is clear that $\lambda = 0$ lies in $\mathfrak{S}(\mathcal{L})$. Precisely 0 is a simple eigenvalue of $\mathcal{L}$ since $\mathcal{M}$ is the unique (up to a multiplication factor) steady state of $\mathcal{L}$.

Combining the above results with Proposition 3.1 leads to a precise description of the spectrum of $\mathcal{L}$:

**Theorem 3.7** *The spectrum of* $\mathcal{L}$ *(as an operator in* $\mathcal{H}$*) consists of the spectrum of* $-\mathcal{L}^-$ *and of, at most, eigenvalues of finite multiplicities. Precisely, setting* $\nu_0 = \inf_{v\in\mathbb{R}^3}\sigma(v) > 0$,

$$\mathfrak{S}(\mathcal{L}) = \{\lambda \in \mathbb{R};\ \lambda \leq -\nu_0\} \cup \{\lambda_n;\ n \in I\}$$

*where* $I \subset \mathbb{N}$ *and* $(\lambda_n)_n$ *is a decreasing sequence of real eigenvalues of* $\mathcal{L}$ *with finite algebraic multiplicities*: $\lambda_0 = 0 > \lambda_1 > \lambda_2 \cdots > \lambda_n > \cdots$, *which unique possible cluster point is* $-\nu_0$.

*Proof* From Proposition 3.4, $\mathcal{L}$ is nothing but a compact perturbation of the loss operator $-\mathcal{L}^-$. Hence, Weyl's Theorem asserts that $\mathfrak{S}(\mathcal{L}) \setminus \mathfrak{S}(-\mathcal{L}^-)$ consists of, at most, eigenvalues of finite algebraic multiplicities which unique possible cluster point is $\sup\{\lambda, \lambda \in \mathfrak{S}(-\mathcal{L}^-)\}$. In particular, up to a rearrangement, one can write $\mathfrak{S}(\mathcal{L}) \setminus \mathfrak{S}(-\mathcal{L}^-) = \{\lambda_n,\ n \in I\}$ with $\lambda_0 > \lambda_1 > \lambda_2 \cdots > \lambda_n \geq \cdots$. We already saw that $\lambda_0 = 0$ since $\mathcal{M}$ is a steady state of $Q$ and $\mathcal{M} \in \mathcal{H}$. Now, since $-\mathcal{L}^-$ is a multiplication operator by the collision frequency $-\sigma(\cdot)$, its

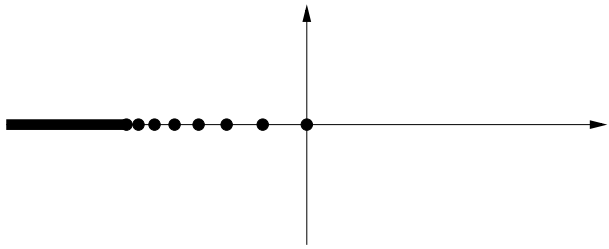

**Fig. 1** Spectrum of the collision operator $Q$ in $\mathcal{H}$

spectrum $\mathfrak{S}(-\mathcal{L}^-)$ is given by the essential range $R_{\text{ess}}(-\sigma(\cdot))$ of the collision frequency. From Corollary 2.5, one sees without difficulty that

$$R_{\text{ess}}(-\sigma(\cdot)) = (-\infty, -\nu_0]$$

where $\nu_0 = \inf_{v\in\mathbb{R}^3} \sigma(v) = \lim_{|v-u_1|\to 0} \sigma(v) = \frac{8}{(2+\mu)^2}\sqrt{\frac{2\pi\vartheta_1}{m_1}}$ is positive. □

*Remark 3.8* We conjecture that, as it is the case for elastic interactions [13], the set of eigenvalues lying in $(-\nu_0, 0)$ is infinite. However, the technical generalization of the proof of [13] appears to be non trivial because of the non zero parameter $\mu$. We thank anyway an anonymous referee for having pointed to us the reference [13].

The above result provides a complete picture of the spectrum of $Q$ as an operator in $\mathcal{H}$ (see Fig. 1) and shows, in particular, the existence of a positive spectral gap $|\lambda_1|$ of $\mathcal{L}$. Note that such a result, combined with Proposition 3.5, has important consequence on the entropy production, since it can be shown in an easy way that the $H$-Theorem reads as

$$\frac{d}{dt}\|f(t) - \mathcal{M}\|_{\mathcal{H}}^2 = \langle \mathcal{L}f(t), f(t)\rangle_{\mathcal{H}}.$$

Consequently, the *Dirichlet form* $B(f) = \langle \mathcal{L}f, f\rangle_{\mathcal{H}}$ plays the role of entropy-dissipation functional and the existence of a spectral gap $|\lambda_1|$ is equivalent to the following coercivity estimate for $B(f)$:

$$B(f) \geq -|\lambda_1|\,\|f\|_{\mathcal{H}}^2 \quad \forall f \perp \text{span}(\mathcal{M}).$$

One deduces easily the following corollary on the exponential trend towards equilibrium:

**Corollary 3.9** *Let $f_0(v) \in \mathcal{H}$ and let $f(t,v)$ be the unique solution to the linear homogeneous Boltzmann equation* (3.2)*. Then, there is some constant $C \geq 0$ such that*

$$\|f(t,\cdot) - \mathcal{M}\|_{\mathcal{H}} \leq C\exp(-|\lambda_1|t)\|f_0 - \mathcal{M}\|_{\mathcal{H}}, \quad \textit{for any } t \geq 0,$$

*where $0 < |\lambda_1| \leq \nu_0$ is provided by Theorem* 3.7*.*

We refer the reader to [19] for details on the matter, and in particular, for an *explicit estimate of the spectral gap* $|\lambda_1|$.

## 4 Application to the Honest Solutions of the Boltzmann Equation

### 4.1 Conservative Solutions

We are interested in this section in applying the result of the previous section to prove the existence of honest solutions to the linear Boltzmann equation for dissipative hard-spheres

$$\partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) = Q(f)(x,v,t), \tag{4.1}$$

with initial condition

$$f(x,v,0) = f_0(x,v) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes dv),$$

where the collision operator $Q$ is given by (1.2). Recall that the streaming operator $T_0$ is defined by

$$\mathcal{D}(T_0) = \{f \in \mathcal{X}, v \cdot \nabla_x f \in \mathcal{X}\}, \quad T_0 f = -v \cdot \nabla_x f$$

where $\mathcal{X} = L^1(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes dv)$. One can define then the multiplication operator $\Sigma$ by

$$\mathcal{D}(\Sigma) = \{f \in \mathcal{X}, \sigma f \in \mathcal{X}\}, \qquad \Sigma f(x,v) = -\sigma(v) f(x,v)$$

where, as in the previous section, $\sigma(v)$ is the collision frequency corresponding to *dissipative hard spheres* interactions and given by (2.5). The following generation result is well-known [3].

**Theorem 4.1** *The operator $T_0$ generates a $C_0$-semigroup of isometries $(U(t))_{t\geq 0}$ of $\mathcal{X}$ given by*:

$$U(t) f(x,v) = f(x - tv, v), \quad t \geq 0.$$

*The operator $A = T_0 + \Sigma$ with domain $\mathcal{D}(A) = \mathcal{D}(T_0) \cap \mathcal{D}(\Sigma)$ is the generator of a contractions $C_0$-semigroup $(V(t))_{t\geq 0}$ given by*

$$V(t) f(x,v) = \exp(-\sigma(v)t) f(x - tv, v), \quad t \geq 0.$$

Let us define now $K$ as the gain operator $Q^+$ endowed with the domain of $A$:

$$\mathcal{D}(K) = \mathcal{D}(A), \qquad Kf(x,v) = Q^+(f)(x,v) = \epsilon^{-2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |q \cdot n| f(x, v_\star) \mathcal{M}_1(w_\star) dw dn.$$

It is clear from (1.5) that, for any $f \in \mathcal{D}(K)$,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (Af + Kf) dx dv = 0, \tag{4.2}$$

while $Kf \geq 0$ for any $f \in \mathcal{D}(K)$, $f \geq 0$. Then, the following generation result is a direct consequence of [1, 25]:

**Theorem 4.2** *There exists a positive contractions semigroup $(Z(t))_{t\geq 0}$ in $\mathcal{X}$ whose generator $G$ is an extension of $A + K$. Moreover, $(Z(t))_{t\geq 0}$ is minimal, i.e. if $(T(t))_{t\geq 0}$ is a positive $C_0$-semigroup generated by an extension of $A + K$, then $T(t) \geq Z(t)$ for any $t \geq 0$.*

The natural question is now to determine whether the "formal" mass conservation identity (1.6) can be made rigorous. Namely, one aims to prove that, for any nonnegative $f \in \mathcal{X}$, the following holds:

$$\|Z(t)f\| = \|f\|, \quad \forall t > 0.$$

The important point to be noticed is the following. If $G = \overline{A+K}$, then any function $\varphi \in \mathcal{D}(G)$ can be approximated by a sequence $(\varphi_n)_n \subset \mathcal{D}(A+K) = \mathcal{D}(A)$ such that $\varphi_n \to \varphi$ and $(A+K)\varphi_n \to G\varphi$ as $n \to \infty$. In particular, (4.2) implies that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} G\varphi\, dxdv = \lim_{n\to\infty} \int_{\mathbb{R}^3\times\mathbb{R}^3} (A+K)\varphi_n dxdv = 0, \quad \forall \varphi \in \mathcal{D}(G).$$

Now, for any given initial datum $f_0 \in \mathcal{D}(G)$, $f_0 \geq 0$, the solution $f(t) = Z(t)f_0$ of (4.1) is such that

$$\frac{d}{dt}\|f(t)\| = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{d}{dt} f(t) dxdv = \int_{\mathbb{R}^3\times\mathbb{R}^3} Gf(t) dxdv = 0,$$

i.e.

$$\|f(t)\| = \|f_0\|, \quad \forall t \geq 0.$$

This means that, if $G = \overline{A+K}$, then the solutions to the linear Boltzmann equation (4.1) are conservative. On the other hand, if $G$ is a larger extension of $A+K$ than $\overline{A+K}$, then there may be a loss of particles in the evolution (see [3] for the matter as well as [2] for examples of transport equation for which such a loss of particles occurs because of boundary conditions). Precisely, if $G \neq \overline{A+K}$ then there exists $f_0 \in \mathcal{X}$, $f_0 \geq 0$ such that

$$\|Z(t)f_0\| < \|f_0\| \quad \text{for some } t > 0.$$

This shows that the determination of the domain $\mathcal{D}(G)$ of $G$ is of primary importance in the study of the Boltzmann equation. This is the main concern of the so-called *substochastic perturbation theory of $C_0$-semigroups* [3].

We point out that the question of the honesty of the semigroup governing the Boltzmann equation also arises in the study of the *space-homogeneous* version of the latter equation. Indeed, it is the unboundedness of the collision frequency (and consequently that of whole collision operator $Q$) that may give rise to *dishonest solutions* to the Boltzmann equation. Actually, to prove the honesty of the $C_0$-semigroup $(Z(t))_{t\geq 0}$, we will adopt the strategy developed first in [1] and systematized in [3]. More precisely, we will show that the gain operator $K$ fulfills the assumption of [1]:

**Proposition 4.3** *There exists $C > 0$ such that, for any fixed $\varrho > 0$,*

$$\operatorname*{ess\,sup}_{|v-u_1|\leq\varrho} \int_{|v'-u_1|\geq\varrho} k(v', v) dv' \leq C.$$

*Proof* Since the kernel $k(v, v')$ differs from the corresponding one for classical gas, except from numerical constants, one can apply *mutatis mutandis* the technical calculations of [1, Sect. 4.1] (see also [3, p. 329–330]) to get the desired estimate. □

As a consequence, one deduces immediately from [1], the main result of this section:

**Theorem 4.4** *The generator $G$ of the minimal semigroup $(Z(t))_{t\geq 0}$ is given by*

$$G = \overline{A + K}.$$

*In particular, the $C_0$-semigroup $(Z(t))_{t\geq 0}$ is honest and*

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} Z(t) f(x,v)dxdv = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v)dxdv \quad \text{for any } f \in \mathcal{X} \text{ and any } t \geq 0.$$

4.2 Consequence on the Entropy Production

As a direct application of the above result (Theorem 4.4), we give a direct rigorous proof of the linear $H$-Theorem of [14]. In order to stay in the formalism of [14], we shall restrict ourselves to the space-homogeneous case. Precisely, let $Y$ denote the set of functions depending only on the velocity and integrable with respect to velocities:

$$Y = L^1(\mathbb{R}^3, dv),$$

equipped with its natural norm $\|\cdot\|_Y$. For any nonnegative $f$ and $g$ in $Y$, we define the *information* of $f$ with respect to $g$ by

$$\mathcal{I}(f|g) = \int_{\mathbb{R}^3} (f(v)\ln f(v) - f(v)\ln g(v))dv$$

with the conventions $0\ln 0 = 0$ and $x\ln 0 = -\infty$ for any $x > 0$. This means that the information is nothing but the entropy functional $H_\Phi$ for the particular choice of $\Phi(s) = s\ln s$. One recalls the main result of [24]:

**Theorem 4.5** *Let $U$ be a stochastic operator of $Y$, i.e. $U$ is a positive operator such that $\|Uf\|_Y = \|f\|_Y$ for any $f \in Y$, $f \geq 0$. Then,*

$$\mathcal{I}(Uf|Ug) \leq \mathcal{I}(f|g)$$

*for any nonnegative $f$, $g$ in $Y$. In particular, if $g \in Y$ is a nonnegative fixed point of $U$ then,*

$$\mathcal{I}(Uf|g) \leq \mathcal{I}(f|g), \quad \forall f \in Y,\ f \geq 0.$$

According to the results of the previous section, it is not difficult to see that the restriction of $(Z(t))_{t\geq 0}$ to $Y$ is a $C_0$-semigroup of stochastic operators of $Y$. Since the unique equilibrium state $\mathcal{M} \in Y$ is space independent, one sees that $(T_1 + K)\mathcal{M} = 0$ and, in particular,

$$Z_Y(t)\mathcal{M} = \mathcal{M}, \quad \forall t \geq 0.$$

Combining this with Theorem 4.5, one obtains a rigorous and direct proof of the $H$-Theorem [14, Theorem 5.1]:

**Theorem 4.6** *Let $f_0 \in Y$ be a given nonnegative (space homogeneous) distribution function with unit mass, i.e. $\|f_0\|_Y = 1$. Assume that $\mathcal{I}(f_0|\mathcal{M}) < \infty$, then*

$$\frac{d}{dt}\mathcal{I}(f(t)|\mathcal{M}) \leq 0 \quad (t \geq 0),$$

*where $f(t) = Z_Y(t) f_0 = Z(t) f_0$ is the unique solution to (4.1) with $f(0) = f_0$.*

*Remark 4.7* Note that, once the conservativity of the solution to the Boltzmann equation asserted by Theorem 4.4, the above $H$-Theorem can be proved by usual standard method of kinetic theory. However, we insist on the fact that such standard proofs require the solution $f(t, v)$ to be conservative and, in some sense, the use of the substochastic semigroup theory.

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